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AMTI

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds " National Mathematical Talent Search Competition " annually and organizes Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education had been instituted by Prof.R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "**The Mathematics Teacher (India)**" and effective from April 1993 is as follows:

Subscription for India*

Category	Individual	Institutional
Life	Rs. 500	Rs. 1000
Annual (Ordinary)	Rs. 50	—
Junior Mathematician - Life	Rs. 250	Rs. 250
Junior Mathematician - Annual	Rs. 25	Rs. 25

The Journal "The Mathematics Teacher" will be supplied free to all members.
Fifty or more subscriptions to Junior Mathematician will entail 20% discount.

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Dr. S.Muralidharan

Editor

EDITORIAL

This issue opens with an interesting article on a geometric inequality by Prithwijit De. In general, Geometry is found to be a difficult topic in the schools. This article presents an interesting generalization of the well known Euler inequality in a triangle. Following this, we present the text of speeches delivered by the invited professionals in the 51st Annual conference of AMTI held at Atul Vidyalaya, Gujarat during December 2016. Professor Parvin Sinclair discussed teaching and learning Mathematics in the open and distance learning mode. This is the age of technology. Using technology in the classroom has been experimented by several teachers. We present an interesting experiment of technology enabled explorations in the class room by Dr Jonaki Ghosh. Professor J Pandurangan discussed the Applications of Matrix theory in his Bhatnagar Memorial lecture and we present an abridged version of his talk. In Professor R.C.Gupta memorial lecture delivered by Dr Devbhadra V Shah, he gave an interesting account of the life of Ramanujan.

It has been observed that the costs of printing and postage have been increasing over time and we find it difficult to meet these

costs. One of the suggestions in the General body meeting of AMTI at Bangalore in December 2017 was to consider sending electronic copy, instead of print copy, of the journal The Mathematics Teacher to the life members. This would reduce the printing and postage expenses considerably. We understand however, that some of our members would still like a print copy of the journal. Hence we are giving an option to the members to opt for electronic copy of the journal. Please see the proposal on page 50 and let us know your option.

Finally we present an account of the Association activities and some photographs from the AMTI conference.

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A Geometric Inequality and its Generalization

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Introduction

A very elegant result in plane geometry due to Euler says that the circumradius R of a triangle is twice as much as its inradius r and equality occurs if and only if the triangle is equilateral. In symbols

$$R \geq 2r. \tag{1}$$

It is quite natural to ask what happens in case of a cyclic quadrilateral with an incircle. That is, if R and r are the circumradius and inradius of the quadrilateral what is the minimum value of R/r. More generally, is it possible to derive the minimum value of R/r for a cyclic polygon with an incircle? The answer is yes and we derive the desired minimum value in this article.

Inequalities for Triangle and Bicentric Quadrilateral

In this section we prove Euler's result for a triangle and also derive the corresponding inequality for a *bicentric quadrilateral* (a quadrilateral having both circumcircle and incircle). Among various ways of proving Euler's result, two proofs stand out. One is algebraic in nature and the other geometric. First let us see the algebraic proof.

Let ABC be the given triangle. Denote by a, b, and c the lengths of the sides BC, CA, and AB respectively. Let $s = \dfrac{a+b+c}{2}$ and let Δ denote the area of ABC. Observe that $s-a$, $s-b$ and $s-c$ are positive quantities. Recall that $r = \dfrac{\Delta}{s}$ and $R = \dfrac{abc}{4\Delta}$. Therefore

$$\frac{R}{r} = \frac{abcs}{4\Delta^2} = \frac{abc}{4(s-a)(s-b)(s-c)}. \tag{2}$$

Now observe that

$$a = 2s - b - c = (s-b) + (s-c) \geq 2\sqrt{(s-b)(s-c)}, \tag{3}$$

$$b = 2s - c - a = (s-c) + (s-a) \geq 2\sqrt{(s-c)(s-a)}, \tag{4}$$

and

$$c = 2s - a - b = (s-a) + (s-b) \geq 2\sqrt{(s-a)(s-b)}. \tag{5}$$

Multiplying these three inequalities leads to

$$abc \geq 8(s-a)(s-b)(s-c) \tag{6}$$

whence

$$\frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)} \geq 2. \tag{7}$$

Equality occurs if and only if $s-a = s-b = s-c$, that is, $a = b = c$, which is same as saying that the triangle is equilateral.

Now the geometric proof. Let ABC be the triangle and let I and O be respectively its incentre and circumcentre. Let AI meet the circumcircle at D. By power of a point theorem

$$AI.ID = R^2 - OI^2. \tag{8}$$

In triangle IBD, $\angle IBD = \dfrac{A+B}{2} = \angle BID$ and thus $ID = DB = 2R\sin\dfrac{A}{2}$. Also $AI = \dfrac{r}{\sin\dfrac{A}{2}}$. Thus from (8)

$$OI^2 = R^2 - 2Rr \tag{9}$$

and as $OI^2 \geq 0$, $R \geq 2r$. Equality occurs if and only if $OI = 0$ which happens only in the case of an equilateral triangle.

If $ABCD$ is a bicentric quadrilateral with $AB = a$, $BC = b$, $CD = c$ and $DA = d$, its circumradius is

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}} \tag{10}$$

where $2s = a+b+c+d$ and its inradius is

$$r = \frac{2\sqrt{abcd}}{a+b+c+d}. \tag{11}$$

Moreover

$$a + c = b + d. \tag{12}$$

Therefore

$$\left(\frac{4R}{r}\right)^2 = \frac{(ab+cd)(ac+bd)(ad+bc)s^2}{(s-a)(s-b)(s-c)(s-d)abcd} \tag{13}$$

which on using $a+c = b+d$ reduces to

$$\left(\frac{4R}{r}\right)^2 = \frac{(ab+cd)(ac+bd)(ad+bc)(a+c)(b+d)}{(abcd)^2}. \tag{14}$$

Now we observe that by AM-GM inequality

$$ab + cd \geq 2\sqrt{abcd};\ ac + bd \geq 2\sqrt{abcd};\ ad + bc \geq 2\sqrt{abcd} \tag{15}$$

and

$$a + c \geq 2\sqrt{ac};\ b + d \geq 2\sqrt{bd}. \tag{16}$$

Therefore

$$\left(\frac{4R}{r}\right)^2 = \frac{(ab+cd)(ac+bd)(ad+bc)(a+c)(b+d)}{(abcd)^2} \geq 32 \tag{17}$$

and hence

$$\frac{R}{r} \geq \sqrt{2} \tag{18}$$

with equality if and only if $ab = cd$, $ac = bd$, $ad = bc$, $a = c$, $b = d$ which collectively imply $a = b = c = d$. The expressions for the circumradius and inradius of a bicentric quadrilateral may be found in [1].

Generalization of the inequality

The methods employed to derive the minimum value of R/r for a triangle and a bicentric quadrilateral do not lend themselves amenably to the computation for a *bicentric polygon*(a cyclic polygon with an incircle). It is quite difficult to obtain closed form expressions for the area, circumradius and inradius of a bicentric polygon having more than four sides. However, we can make use of trigonometry to good effect to express the circumradius and inradius of a bicentric polygon. As we shall see, this switchover to trigonometry will prove to be a success. Let Ω be a circle with centre O. Let $A_1, A_2, \ldots, A_n$ be n points on Ω with $n > 4$ such that

$$\angle A_i O A_{i+1} = 2\alpha_i \tag{19}$$

for $i = 1, 2, \ldots, n$ and $A_{n+1} = A_1$. Then

$$a_i := A_i A_{i+1} = 2R \sin \alpha_i \tag{20}$$

where $A_{n+1} = A_1$ and

$$r = \frac{2[A_1A_2\ldots A_n]}{\sum\limits_{i=1}^{n} a_i} = \frac{R}{2}\left(\frac{\sum\limits_{i=1}^{n}\sin 2\alpha_i}{\sum\limits_{i=1}^{n}\sin\alpha_i}\right) \tag{21}$$

where $[A_1A_2\ldots A_n]$ denotes the area of the polygon. Observe that $\sum\limits_{i=1}^{n}\alpha_i = \pi$. From (21)

$$\frac{R}{r} = \frac{2\sum\limits_{i=1}^{n}\sin\alpha_i}{\sum\limits_{i=1}^{n}\sin 2\alpha_i}. \tag{22}$$

Note that in the denominator of the expression on the right hand side of (22) at most one term may be negative. This is because at most one of the α_i's can lie in the interval $(\pi/2, \pi)$. Thus

$$\frac{R}{r} = \frac{2\sum\limits_{i=1}^{n}\sin\alpha_i}{\sum\limits_{i=1}^{n}\sin 2\alpha_i} \geq \frac{2\sum\limits_{i=1}^{n}\sin\theta_i}{\sum\limits_{i=1}^{n}\sin 2\theta_i} \tag{23}$$

where $\theta_i \in (0, \pi/2]$ and $\sum\limits_{i=1}^{n}\theta_i = \pi$. Without loss of generality let

$$0 < \theta_1 \leq \theta_2 \leq \cdots \theta_n \leq \frac{\pi}{2}. \tag{24}$$

Then

$$0 < \sin\theta_1 \leq \sin\theta_2 \leq \cdots \leq \sin\theta_n \leq 1 \tag{25}$$

and

$$1 > \cos\theta_1 \geq \cos\theta_2 \geq \cdots \geq \cos\theta_n \geq 0. \tag{26}$$

Therefore by Chebyshev's inequality

$$n\sum_{i=1}^{n}\sin\theta_i\cos\theta_i \leq (\sum_{i=1}^{n}\sin\theta_i)(\sum_{i=1}^{n}\cos\theta_i) \tag{27}$$

and upon rearrangement this yields

$$\frac{2\sum\limits_{i=1}^{n}\sin\theta_i}{\sum\limits_{i=1}^{n}\sin 2\theta_i} \geq \frac{n}{\sum\limits_{i=1}^{n}\cos\theta_i}. \tag{28}$$

The cosine function is concave on $(0, \pi/2]$. Therefore by Jensen's inequality

$$\frac{\sum\limits_{i=1}^{n}\cos\theta_i}{n} \leq \cos\left(\frac{\sum\limits_{i=1}^{n}\theta_i}{n}\right) = \cos\frac{\pi}{n}. \tag{29}$$

Reverting to (23) and (28) we obtain

$$\frac{R}{r} \geq \frac{2\sum\limits_{i=1}^{n}\sin\theta_i}{\sum\limits_{i=1}^{n}\sin 2\theta_i} \geq \frac{n}{\sum\limits_{i=1}^{n}\cos\theta_i} \geq \frac{1}{\cos\frac{\pi}{n}}. \tag{30}$$

Therefore

$$R \geq r\sec\frac{\pi}{n} \tag{31}$$

with equality if and only if $\alpha_i = \frac{\pi}{n}$ which implies

$$a_1 = a_2 = \cdots = a_n$$

Thus the polygon is equilateral. But as it is cyclic, equality of sides implies equality of angles. Therefore equality is achieved when the polygon is regular. It is easy to verify that the values of the lower bound for a triangle and a bicentric quadrilateral are in agreement with what we obtained earlier by way of more explicit calculations. It has to be said that the latter method can be easily adapted to prove the inequality for a triangle and a bicentric quadrilateral.

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Teaching and Learning Mathematics through the ODL Mode

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Good morning, and thank you for inviting me to share my understanding with you on an area which I have spent nearly half my life studying and practicing.

Let me, start by a brief reflection on what mathematics is. Is it the science of quantity, as Aristotle put it? Is it the science of pattern, as Devlin et al put it? Or is it the science of structure and relationships, as Courant & Robbins put it(ref.[1])? To me, it is primarily a certain way of thinking and developing new ideas that are consistent with the concepts/processes developed earlier. Consistency has a certain connotation here, as we know. It requires working with axioms that are in place, and providing evidence for accepting any statement as valid. This evidence has to be in line with the logical processes that are particular to mathematics. Herein lies the central idea of mathematics,viz., proof, which is built around inductive, deductive and abductive reasoning.

Therefore, it is these ways of reasoning and proving that need to be the crux of the teaching and learning of mathematics.

But, is this what is transacted in the mathematics classrooms

Theme talk in the 51st Annual Conference of the Association of Mathematics Teachers of India (AMTI)

of our schools, colleges, PG programmes? Rarely so. Teaching is reduced to definitions, statements and proofs/solutions being given in the banking model to the learners, who are expected to accept it unquestioningly. Of course, questions/problems are given, to be solved in the same way as the teacher noted on the board, perpetuating the myth that there is a single solution to a given problem. The culture of giving notes to students, and making them do many many problems of exactly the same kind, has changed very little over the decades. And hence, the fear and awe of mathematics has not reduced over centuries!

Here is where a good Open and Distance Learning (ODL) system can help, and has helped. Briefly, this has happened because this system is learner-centred, with the curriculum design being learner-oriented and the learning materials being designed, created and delivered using the best resources available, not just in one institution, but across the state/country/world. Let me iterate the expectations the National Policy of Education (NPE), modified 1992(see[2]), has from such a system

The open learning system has been initiated in order to augment opportunities for higher education, as an instrument of democratising education and to make it a lifelong process. The flexibility and innovativeness of the open learning system are particularly suited to the diverse requirements of the citizens of our country, including those who had joined the vocational stream.

The Indira Gandhi National Open University, established in 1985 in fulfilment of these objectives, will be strengthened. It would also provide support to establishment of open

universities in the states.

The National Open School will be strengthened and open learning facilities extended in a phased manner at the secondary level in all parts of the country. To understand the expectations and the claim I made about the system, let me first tell you how an ODL system functions. The O, denoting open, allows for flexibility in the rigidity several structures of the conventional system have. For instance, in eligibility criteria, course combinations, types of programmes offered, age limits, pace of completing programmes, mode of transaction, etc. The DL in ODL refers to distance learning, which is actually one of the facets of the O mentioned earlier. Here the transaction takes place through various non-human media primarily, like print, DVDs, web portals, mobiles. However, the human element is very much present, but not directly visible! The ODL system has been a boon across the world for a large category of learners, including the marginalisedwomen who cant leave their home, people with disabilities, people who didnt get a chance to study in the normal course, people who decide to update themselves, and so on. An ODL learner enrolls in a programme online or through a centre nearest to her. Once she is admitted to the programme, she is informed of what the course package comprises, the cohort of students she is part of, and how the mentors will support her, either online synchronously and asynchronously, or through face-to-face sessions at a study centre nearest to her usually at a time convenient to the mentor and the cohort.

Coming to the systems in India, the major players are the National Institute of Open Schooling (NIOS) and the Indira

Gandhi National Open University (IGNOU), at the school & tertiary levels, respectively. Many states have their own equivalents (11 SOSs and 13 SOUs), and there are Directorates of Distance Education in most universities. The primary medium of transaction used in all these is the print medium, with academic counselling through face-to-face sessions at study centres as well as through e-mail (in some cases). These sessions are usually not compulsory to attend. All programmes are credit based, with IGNOU, for example, defining a credit as 30 hours of student study time on an average (note that this is NOT determined by teaching time!). Most courses are 4 or 8 credits worth, which approximate a 1-semester or a 1-year course, respectively, in the conventional system. Each programme comprises several such courses. The programmes are designed, developed and delivered with the help of human resource and infrastructure available in the conventional system across the country, thus making effective use of the nations resources. As far as the core faculty in any of these institutions is concerned, it is very small, in fact, some say, too small.

IGNOU offers most of its programmes in Hindi and English medium, and allows learners ample time to complete the programmes, e.g., the B.Sc can be completed in a minimum of 3 years, and a maximum of 8 years. The learners are sent printed blocks in the medium asked for. The difference between an IGNOU block and a textbook is akin to that between red and white. Both seek to share content, but IGNOU material does so in a very different manner– it allows for context, interactivity, dialogue with the learner, and several points built in for self-assessment, etc. An IGNOU block is more like a facilitator. This is meant completely for self-learning. Of

course, as there is a gap between the ideal and reality, it requires further supplementary support in the form of academic counsellors, audio/video DVDs, online links, further reading suggestions. An assignment, which is more like an open book test, acts as a further learning tool and contributes to the overall evaluation as well. How is it a learning tool? The responses are evaluated by the local counsellor, who writes down her positive hints/comments/motivating remarks, and the marks given. This assignment has to be submitted before the learner fills up her exam form. The learner should also get it back before the exam so she can see her strengths and weaknesses, and learn from the tutor comments.

There are several popular misconceptions about this mode that need to be corrected. One is that an ODL learner learns by correspondence alone, thus meaning that she has no human teaching support. Most ODL programmes have 10% or more time allotted for face-to-face contact with academic tutors/counsellors at centres across their jurisdiction. In fact, in this day of technological growth, most countries offer online courses with synchronous and asynchronous contact with the mentor as well as peers. We all have heard of MOOCs(massive open online courses),and the MHRD is also pushing for MOOCs to be developed at the certificate, diploma and degree levels. Several courses are available online(e.g., see [3],[4]).

Yet another misconception is that skills cannot be learnt /taught through this mode. Those who understand open learning believe there is nothing that can be taught in the conventional system that cannot be learnt through the ODL system. This belief is borne out by experience. For instance,

several teacher education programmes are offered through ODL. Let me tell you about three in particular–the Certificate Programme in Teaching of Primary School Mathematics (CTPM), B.Ed and Ph.D(Math Education) offered by IGNOU.

As the name suggests, CTPM is a 6-month(minimum period) programme aimed at any adult who wants to teach mathematics, to a child or children, at the primary school level. Its objective is to help the learner understand how children learn mathematics, and practise teaching mathematics accordingly. It includes a project component through which the learner is required to actually engage with a large number of children in the process of learning different areas of mathematics. For help in undertaking this component, a Project Guide is sent along with the other blocks (printed course materials). Further, the counsellor at the study centre is required to spend time with each learner for discussing her proposal and guiding her through her project. The counsellor also has to give a certificate saying that the work of the learner is bona fide.

CTPM is built around the processes involved in mathematics and in constructivism. Its aim is to improve the skill of teaching mathematics. The assessment component of this, as of most programmes offered by IGNOU, is of four kinds – self assessment, assessment of and by the counsellor (for those who attend the counselling sessions), the assignment component and the term-end assessment and evaluation. Apart from this, the project component, in the form of a project report, is evaluated by an evaluator drawn from a panel of evaluators from across India, approved by the Programme Team and the School

Board as authorised by the Academic Council of IGNOU. For completing the programme,the learner needs to pass separately in the assignment and the term-end examination of each course of the programme, as well as in the project component. Next, the B.Ed programme, which is of two years. is aimed at adults who have done a Diploma in Education in the conventional system, as per the current NCTE norms. Admission is through an entrance test, and only 2500 learners are taken every year, a recent cap enforced by the NCTE on every OU. The B.Ed is delivered as the other programmes are delivered. Further, intensive and extensive contact programmes are held regularly at centres across the country, where experts assess the quality of learning and help to improve this quality. Internship in schools is also an essential component of this programme.

The Ph.D(Math Education) was started some years ago, and follows the prevailing UGC norms. This is an area of research that most universities do not offer. Here too support is provided to nurture the learners abilities and skills required to undertake research. Compulsory coursework comprises specially designed courses on research methodology as well as on evolution of mathematics and theories of mathematics education.

Thus, skills of various kinds and at various levels can be nurtured through the ODL. Coming to the schooling system, the NPE requires Science and mathematics should be an integral part of general education till the end of the school stage., i.e., everyone to study mathematics till Class X. This was because, again quoting from the NPE, Mathematics should be visualised as the vehicle to train a child to think, reason, analyse and to articulate logically. Apart from being a specific

subject, it should be treated as a concomitant to any subject involving analysis and reasoning. Thus it is seen as a vehicle to develop certain thinking processes, including questioning, evidence based acceptance of facts, critiquing, analyzing, etc., in the people of this country. However, as noted earlier, though mathematics is taught, it is not learnt.

The mathematics curriculum, as laid down in NCF 2005(see [5]), has tried to bring to the fore what maths is about, briefly noted as the mathematisation of the childs thinking. However, the actual classroom transaction has not changed very much. The assessment and evaluation methods have also not changed, though CCE has supposedly come into currency.

The NIOS does not insist on mathematics and science being learnt till Class X (see[6]). In fact, a learner picks any five subjects out of a long list, of which one or two languages out of 17 are chosen, and mathematics is a choice against Science and Technology, Painting, Data Entry Operation, Social Sciences ,etc. However,61.5% of learners(239163 in 2012-13) opt for mathematics, and the passing rate is about 75% . At the Senior Secondary stage, about 30% take mathematics, with a passing rate of 65%. About 20% of these learners are female. The learning is done in much the same way as mentioned above, with an assignment required to be submitted, which has 20% weightage towards the final grade. Finally the learner sits for an exam, which is held twice a year, and the learner has 5 years to complete the courses in. Let me now come to the ways we can use in ODL to nurture the mathematisation of our learners. One way of activating required thought processes in our mathematics learners is to focus on why a certain statement

is true or not, which is what IGNOU does. For instance, be it in the exercises in the material.or in the assignment.or the term end exam,if we have discussed Sets we can ask Is { x — x ¡ 8} a set? Why? Or, is {3, , Mars} a set? Why, or why not?.

Again, rather than giving the definition of a relation, say, you will find it being led into through several examples and non-examples, building towards what they may then expect as a definition. We then ask them for more examples and non-examples, may be from situations they have worked on earlier, or from life around them. In fact, it is the counsellor who can help localise the learning to the learners context, since she is best placed for doing so.

Yet another device we use is to give proofs, and ask if they are correct, or not, and why.

The ODL material is often peppered with historical references and relevant anecdotes, that allow the learner a glimpse into how the concept concerned could have arisen. Further, the material unfolds in units of learning called sections, which are made small enough for a learner to study them in a sitting at a time. The solutions/comments on the exercises are added at the end of each unit, to allow the learner to assess whether her understanding is correct or notremember, in India she usually has no one else to discuss this with till the next counselling session. (IGNOU allows for 5 counselling sessions for a semester course.)

Regarding the learning outcome, though no formal study has been undertaken, from feedback taken from peers and learners we have found that good IGNOU mathematics undergraduates

are as good or better in their mathematical abilities than those in the conventional system usually. Even the average IGNOU learners are better than the average conventional system learner. And this is not surprisingthey have developed the ability of independent study, which is crucial for life-long learning. However, only about 20-30% IGNOU learners pass out of the first level courses on Elementary Algebra and Calculus, where passing requires a minimum of 35% in each component, as noted earlier. About 10% pass the Abstract Algebra and Analysis courses. Regarding the downside for the ODL system, since it functions as a system, all its parts need to be up to par for this to work well. For instance, if the counsellor support is lacking or materials do not get to the learners in time, there is a huge problem. The learners can get de-motivated or move to substandard material in the form of keys/guides published by unscrupulous persons. Even when counsellors are available, as for the calculus course, IGNOU learners have often taken to keys,and have failed the exam because what is taught in those books are often wrong. Let me end on an optimistic note about the systemIGNOU math learners have joined institutions like JNU and IIT for higher studies in mathematics. Many went into successful careers in the computer world. Some joined the civil services too! Thank you.

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Developing Students Mathematical Thinking through Technology Enabled Explorations

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Abstract: This paper highlights the role of technology in mathematics education with evidence of students work in senior secondary school. Examples from three research studies will be cited where students explored various applications of mathematics using graphics calculators and spreadsheets. Technology enabled explorations helped students to develop a deeper insight into important mathematical ideas, provided access to higher level concepts and played a key role in developing their mathematical thinking. Through these studies, we will illustrate that the considered use of technology in a traditional mathematics classroom can help to create a rich and motivating environment for learning mathematics.

Keywords: Mathematical thinking, technology enabled explorations, graphics calculators, spreadsheets

Introduction

The last few decades have witnessed a major shift of paradigm as far as mathematics teaching and learning is concerned, in many countries across the world. Mathematics education is being revolutionized with the advent of new and powerful technological tools in the form of dynamic

Prof. A Narasinga Rao Memorial Lecture at 51st Annual Conference Association of Mathematics Teachers of India (AMTI)

geometry software (DGS), computer algebra systems (CAS), spreadsheets and graphic calculators which enable students to focus on exploring, conjecturing, reasoning and problem solving. Some research studies such as the ones by Heid (1988) have shown that the appropriate use of technology can lead to better conceptual understanding and make higher level processes accessible to students without compromising on paper pencil skills. In another study, Heid (2001) describes CAS as a cognitive technology which makes higher level mathematical processes accessible to students. Kissane (2008) describes some pedagogical affordances of the graphics calculator in visualizing and exploring concepts in calculus. According to him, procedural competence can be best developed when students understand the underlying ideas well (p.16). Laborde (2002) suggests that learning geometry with a Dynamic Geometry Environment (DGE) might offer students possibilities to construct and manipulate geometrical figures and do empirical investigations. Leung (2012) describes that a key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities (p.2). Thus mathematics education researchers across the world have conducted many studies which support the use of technology in mathematics teaching and learning. In India, however, hardly any research studies have been conducted to study the impact of integration of technology in mathematics education. In this paper, we shall highlight that the appropriate use of technology in mathematics classrooms can significantly and positively impact mathematics learning. We shall describe some aspects of a few research studies where technology helped to promote visualization, exploration and discovery of concepts

through the multi-representational approach. In section 2 we shall describe how grade 11 students, while exploring fractal constructions through spreadsheets, developed algebraic thinking. Section 3 will highlight how grade 12 students used graphics calculators and spreadsheets to explore the concept of chaos. Section 4 will focus on the role of graphics calculators in enabling grade 12 students to explore application of matrices.

Exploring fractal construction through spreadsheets

Generalization as a skill is fundamental to mathematics learning and is required at various stages of the school mathematics curriculum. Research describes two kinds of generalization (Kinach, 2009), namely, generalization by analogy and generalization by extension. Generalization by analogy refers to observing a pattern, extending a sequence to the next few terms and being able to relate a particular term of the sequence to its previous terms. This kind of generalization requires recursive thinking. Generalization by extension, on the other hand, refers to writing a formula for the nth term of a sequence and this requires explicit thinking. Both kinds of generalization require abstraction and form the core of algebraic thinking. In this section we shall highlight that the topic of fractals provides an authentic context for engaging in recursive as well as explicit thinking. According to Kinach (2009) *Calculating the area and perimeter of the growing pattern of red triangles in the famous Sierspinski triangle and then expressing a formula for the area and perimeter for any iteration ... are more advanced examples of generalization by analogy and extension.* (p.443)

Being able to work with multiple representations such as tables, pictures, graphs and abstracting their interrelationships are an essential aspect of developing algebraic reasoning. We shall describe a part of a module where 30 grade 11 students explored the topic of fractals. Prior to this activity they had studied geometric sequences and were familiar with the formulae,

$$S_n = \frac{a(1-r^n)}{1-r}, \quad S_\infty = \frac{a}{1-r}, \quad |r| < 1$$

The primary goal was to engage them in exploring various patterns within fractal constructions through pictorial, tabular, symbolic and graphical and to make connections between these representations. They used MS Excel to explore the growth of fractals at higher stages. We will highlight that through the spreadsheet exploration they developed an insight into the nature of fractal geometry and engaged in meaningful generalization tasks emerging from the construction process.

Pictorial representations

In the very first session of the module, students were introduced to the Sierspinski triangle construction by the author (who was also their teacher. An equilateral triangle (stage 0) was cut out from a sheet of paper. The mid-points of the sides were joined, to obtain four smaller triangles and the center triangle was removed. This piece with a triangular hole was referred to as stage 1. Students observed that stage 1 comprised three identical smaller copies of stage 0 (each copy was a smaller equilateral triangle). The process of creating smaller equilateral triangles and removing the center triangle was repeated to obtain subsequent stages. Figure 1 shows stages 0, 1 and 2 as obtained by a student. The white triangular portions represent

the triangular holes.

After the construction process was over, some time was spent

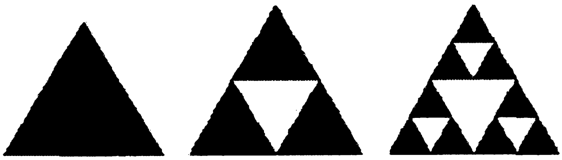


Figure 1: Stages 0 to 2 of the Sierspinski triangle as depicted by a student

on discussing students observations. A few students said that the process could go on forever although many could not describe what higher stages would look like. One student commented that the number of triangular holes will go on increasing referring to the parts which are being removed. Majority of students agreed that the number of triangles will increase at every stage and will also get smaller in size. To give a direction to their thinking, students were assigned two tasks. The first task required them to count the number of shaded triangles in stages 0 to 3 and predict the number for stages 4 and 5. They were also required to find a rule for the number of shaded triangles at the n th stage. In the second task, they had to find a rule for shaded area at the various stages and also at the nth stage (given that the area of the equilateral triangle at stage 0 is 1 square unit). At this point the teacher helped students to make the observation that stage 1 has three smaller copies of stage 0. Similarly stage 2 has three smaller copies of stage 1 and nine still smaller copies of stage 0. This idea of identifying smaller copies of previous stages in subsequent

stages was introduced as self-similarity. Figure 2 was used by the teacher to explain this idea.

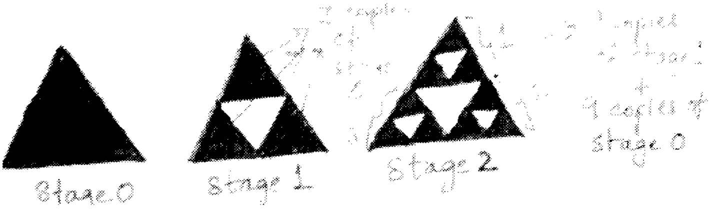


Figure 2: The idea of self-similarity finding scaled down copies of previous stages in a given stage

Tabular representations

Task 1 was easily done by all students as they observed that the number of shaded triangles at each stage was a power of 3 and using a multiplying factor of 3, they came up with the geometric sequence $1, 3, 3^2, 3^3, \ldots$ However, the second task posed a challenge for a few students. While they concluded that the shaded area at stage 1 is $\frac{3}{4}$ units (since only three of the 4 smaller equilateral triangles were shaded), they were unable to extend the idea to stage 2. A few students pointed out the shaded area at stage 1 is being divided into 4 equal parts in stage 2 and one of these parts is being removed thus leading to them to conclude that the shaded area at stage 2 is $\frac{3}{4}$ of $\frac{3}{4}$ that is, $\frac{9}{16}$ or $\left(\frac{3}{4}\right)^2$. This idea was taken up by others and extended to the fact that the multiplying factor in the sequence of shaded areas was $\frac{3}{4}$. Finally a majority of the class obtained the geometric sequence

$$1, \frac{3}{4}, \left(\frac{3}{4}\right)^2, \left(\frac{3}{4}\right)^2, \ldots$$

to represent the shaded area at various stages. Being able to write subsequent terms of the geometric sequences may be considered as examples of generalization by analogy. However, finding the formula for the n th stage entails generalization by extension. This required them to observe that the exponents of 3 and $\frac{3}{4}$ in the two sequences coincide with the stage number. With facilitation, students were able to conclude that the n th terms of the sequences were 3^n and $\left(\frac{3}{4}\right)^n$ respectively.

Symbolic representations recursive versus explicit reasoning

At this stage, the teacher tried to help students make connections between their recursive and explicit reasoning. She introduced the following symbols and asked them to write the n th terms of the two sequences using these $S_n =$ number of shaded triangles at stage n, $A_n =$ Shaded area at stage n. Students had to relate the formula of stage n with that of stage $n-1$ for both sequences. Writing the recursive relation $S_n = 3 \times S_{n-1}$ for the number of shaded triangles, took some scaffolding. The teacher had to emphasize that $S_1 = 3 \times S_0$ and $S_2 = 3 \times S_1$ to help them see the relation. For shaded area, students came up with the recursive formula, $A_n = \left(\frac{3}{4}\right) \times A_{n-1}$ more easily. At this point, the teacher asked them to express the self-similarity of the Sierspinski triangle using the same ideas. After some facilitation, many students could articulate the idea that stage n has three copies of stage $n-1$, 9 copies of stage $n-2$ etc. For the teacher, this was a high point of the class, as it convinced her that students had succeeded in generalizing the Sierpinski triangle construction through multiple representations.

A spreadsheet exploration

Finally students were assigned the task of describing what would happen as n, the number of stages, approached infinity. They conjectured that the number of shaded triangles would become very large and some used the phrase will approach infinity. For shaded area, many students said it would get lesser and lesser. To help them visualise this numerically, the students were encouraged to explore these sequences on MS Excel by generating values up to stage 20 (see Figure 3). For example, the first column shows n the stage number (up to 20), second column, the number of shaded triangles, which was obtained by entering 1 in the first cell (say C3) and $= C3*3$ in cell C4. The sequence of shaded areas was similarly obtained in the third column. Graphing the sequences revealed that the number of shaded triangles was growing exponentially whereas the shaded area was approaching 0. Thus Excel played a pivotal role in helping students visualize the fractal construction process, numerically and graphically. Students explorations took an

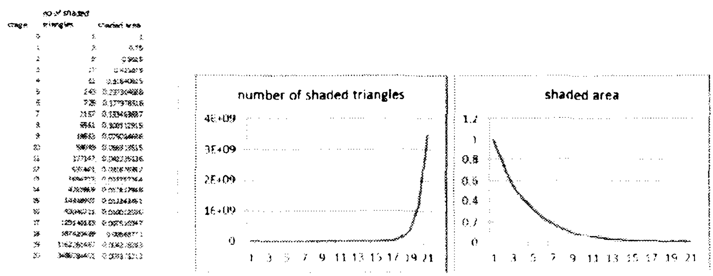


Figure 3: Numerical and graphical representations of the geometrical sequences arising from the Sierspinski triangle construction in MS Excel

interesting turn at this stage in the module. They wanted to

know what would happen if, they took the sum (to infinity) of the geometric progression

$$1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots$$

The teacher encouraged them to explore the cumulative sum of shaded areas represented by the above progression in MS Excel. Figure 4 shows the Excel output where the first column

stage	shaded area	sum of area upto stage n
0	1	1
1	0.75	1.75
2	0.5625	2.3125
3	0.421875	2.734375
4	0.31640625	3.05078125
5	0.237304688	3.288085938
6	0.177978516	3.466064453
7	0.133483887	3.59954834
8	0.100112915	3.699661255
9	0.075084686	3.774745941
10	0.056313515	3.831059456
11	0.042235136	3.873294592
12	0.031676352	3.904970944
13	0.023757264	3.928728208
14	0.017817948	3.946546156
15	0.013363461	3.959909617
16	0.010022596	3.969932213
17	0.007516947	3.97744916
18	0.00563771	3.98308687
19	0.004228283	3.987315152
20	0.003171212	3.990486364

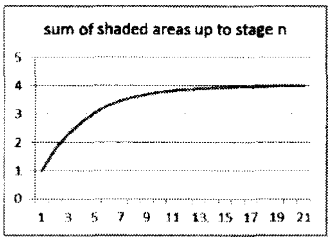


Figure 4: Numerical and graphical representations of the cumulative sum of shaded areas of the various stages of the Sierspinski triangle in MS Excel

represents the stages, the second column, terms of the sequence of shaded areas, and third column, the cumulative sum of areas. Indeed by the 20th term the cumulative sum approaches a fixed value, 4! This was verified by students using the formula $S_\infty = \frac{a}{1-r}$, where a is the initial term and r, the common ratio of the geometric progression. Thus $S_\infty = \frac{1}{1-\frac{3}{4}} = 4$. While the sequence of shaded areas was approaching 0 as n approached infinity, the cumulative sum of areas was approaching 4. The graphical representations led to an interesting discussion in the class. But how can we explain the infinite process leading to a

fixed value? some students asked. Another group of students, after some discussion, explained the amount of area getting added at each successive stage is decreasing, so effectively, the total area is approaching a fixed value. In the beginning of the module students had relied more on pictorial and tabular representations to understand the fractal constructions, but later they made the transition to representing the same ideas using symbols, thus obtaining recursive and explicit formulae. A similar exploration of the Sierspinski triangle was done by pre-service teachers in their mathematics course of their teacher education programme. A detailed description of the same may be found in Ghosh (2016).

Understanding chaos using graphics calculators

A group of grade 12 students had taken up the project of exploring the chaotic behaviour of functions. These students had studied the topic of functions in their regular classes and were familiar with linear, quadratic, cubic, exponential and logistic functions. In order to explore the idea of chaos, they began with the logistic function $f(x) = ax(1-x)$ and explored the iterations of the function for

1. $a = 3.3$ and taking initial values $x_0 = 0.1, 0.6, 0.7, \ldots$
2. $a = 3.8$ and taking initial values $x_0 = 0.2, 0.21, 0.22, \ldots$

Graphics calculators were used as the primary vehicles of exploration. In order to explore the given task, students used the recursive mode of the calculator (RECUR) where the function was entered as a recursive sequence, $a_{n+1} = 3.3a_n(1 - a_n)$. Thirty iterations of this sequence were generated in the TABLE mode and were also graphed. The connected graph for

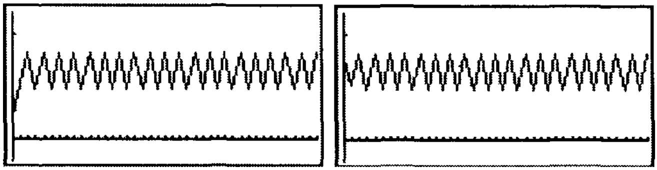


Figure 5: Iterations of the recursive sequence $a_{n+1} = 3.3a_n(1-a_n)$ for $x_0 = 0.1$ and 0.6

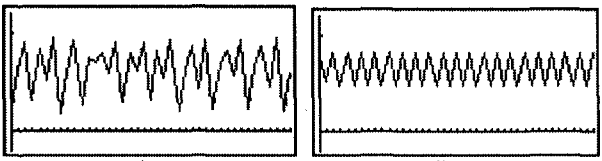
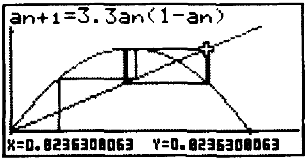


Figure 6: Iterations of the recursive sequence $a_{n+1} = 3.8a_n(1-a_n)$ for $x_0 = 2.1$ and 2.2

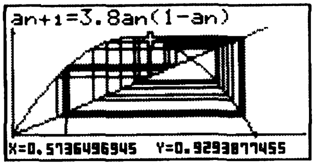
$a = 3.3$ for $x_0 = 0.1$ and 0.6, shown in Figure 5, reveals a powerful progression of the terms.

Students took different starting points and concluded that the graphs settle down into the same pattern even though they may look different in the beginning. However for $a = 3.8$ a very different scenario emerged. Different initial values x_0 led to different graphical patterns. Figure 6 shows the connected graphs for $x_0 = 2.1$ and 2.2. Students articulated in their observations that even though the initial values are very close, the graphs become very different rather quickly and They dont appear to settle down. This situation was used by the teacher to introduce the idea of sensitive dependence on initial conditions.

The graphics calculator provides another powerful way of vi-



(i)



(ii)

Figure 7: The cobweb plots for $a = 3.3$ and $a = 3.8$

sualizing recursive sequences using cobweb plots. This feature was exploited by students to see the long term behaviour of various recursive sequences. Figure 7 shows the cobweb plots for $a = 3.3$ and $a = 3.8$ For $a = 3.3$, the plot shows a clear convergent rectangular pattern whereas for $a = 3.8$ the pattern is rather chaotic. Sometimes a convergent pattern seems to appear but it disappears very quickly.

In this project graphics calculators played a crucial role in helping students visualize the notion of a chaotic procedure. Although numerical and graphical explorations of recursive sequences can be done on a spreadsheet, like MS Excel, the additional feature provided by the graphics calculator of creating cobweb pots on the graphics calculators was particularly helpful in developing a graphical understanding of the iterative process.

Exploring applications of matrices

In this section we shall describe how 32 grade 12 students explored the Hill Cipher method (Eisenberg, 1999), an application of matrices to cryptography, as a part of a module on Modelling and Applications designed by the author. Students were familiarised with some preliminary information related to ciphers as well as basic concepts of modular arithmetic. After this they were introduced

to the Hill Cipher method and were given 29 characters along with their numerical values as shown in Table below:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P	Q	R	S	T	U	V	W	X	Y	Z	.	_	?	
15	16	17	18	19	20	21	22	23	24	25	26	27	28	

The encoding matrix or key was chosen as $E = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ The encoding process (converting the plaintext to the ciphertext) was explained through the following steps.

Step 1. Let the plaintext message be $HOW\_R\_U?$

Step 2. Convert it to its substitution values from the substitution table and group them in pairs

$$7 \quad 14 \quad 22 \quad 27 \quad 17 \quad 27 \quad 20 \quad 28$$

Each pair will form a column of the message matrix M.

Step 3. Compute the product EM and reduce it modulo 29.

$$\begin{aligned} EM &= \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 7 & 22 & 17 & 20 \\ 14 & 27 & 27 & 28 \end{pmatrix} \\ &= \begin{pmatrix} 35 & 76 & 71 & 76 \\ 119 & 255 & 240 & 256 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 18 & 13 & 18 \\ 3 & 23 & 8 & 24 \end{pmatrix} \quad \text{mod } 29 \end{aligned}$$

Step 4. The pairs of numbers 6 3 18 23 13 8 18 24 lead to the encrypted message or ciphertext is $GDSXNITY$. This is an example of a Hill-2-cipher as it uses 2×2 matrices. Some students used the graphics calculator for the matrix computations while others used MMULT and MINVERSE commands in Excel. Figure 8 shows screenshots of the encoding

process on Excel and Figure 9 shows similar computations on the graphics calculator.

To decode the cipher text the inverse of the matrix $E = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$, that is $E^{-1} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$ is required. The details of this method may be found in Ghosh (2015).

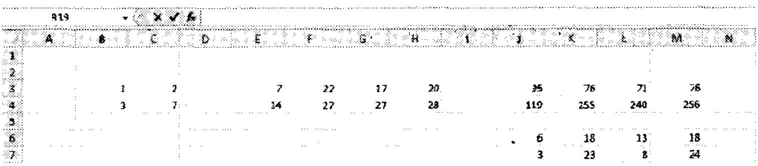


Figure 8: An Excel screenshot of the encoding process as done by the students

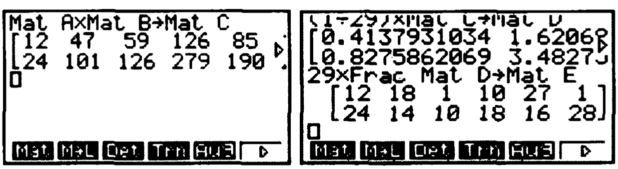


Figure 9: Screenshots of the encoding process as done by the students on the Casio 9860 G II

Students explorations

Students found this technique very exciting. They wanted to try out examples of Hill-3 ciphers. For many, it was a challenge to come up with 3 by 3 matrices with determinant 1. However with help from other students they managed to find some examples. Students were divided into groups of three. Out of the ten groups, six were able to work with Hill-3 ciphers. Two

groups ventured into Hill-4 ciphers and the rest perfected their work with Hill-2 ciphers. After this they designed a cipherquiz where each group was required to encode a message and post it on the whiteboard after declaring the key. The receivers had to use the inverse key to decode the message. Students were asked to choose encoding matrices with determinant equal to 1. All students performed the computations using technology. By the end of the fifth session, all 32 students were able to work with Hill ciphers. After this, students began to come up with queries – what if the determinant of the encoding matrix is not 1? Can anyone crack the Hill cipher? In fact one student asked whether it is possible for a person (other than receiver or sender) to crack the code, that is, figure out the encoding matrix if the plaintext and ciphertext are known. This was undoubtedly the high point of the module for the author! This created the context to introduce the students to Cracking theorem. It was really satisfying when some students took up the challenge of writing a C++ program for the Hill Cipher method. During this attempt, they used their knowledge of matrix theory and logical thinking to obtain the desired output. One student took up a project of exploring the possibility of modifying the Hill Cipher method using the concept of periodic matrices.

Conclusion

In this paper the author has attempted to highlight the role of technology in enabling students to visualize and explore important mathematical concepts through the multi-representational approach. Evidence of students work from three research studies reveal, that if properly used, technology can significantly

and positively impact mathematics learning. In the first study, students engaged in recursive and explicit thinking while attempting generalization tasks arising out of fractal constructions. They developed an insight into fractal geometry as they explored geometric sequences using multiple representations. While they used their pictorial representations to explore the iterative growth, spreadsheets helped to visualize the process numerically and graphically thus leading to a big picture understanding of the fractals. In the second study, graphics calculators helped students to explore the concept of chaos. They used the graphing and recursive mode of the calculator to observe the behaviour of the logistic function numerically, graphically and symbolically. In the third study, graphics calculators and Excel took over lengthy matrix calculations and enabled students to explore the Hill Cipher method, an application of matrices to cryptography. In all the examples cited in this paper, students were able to see the relevance of mathematics to practical problems. Further, technology played the role of an amplifier (Heid, 2000) giving them access to higher level mathematical concepts. Technology helped to facilitate a multi-representational approach to learning concepts and this supports Heids (2001) theory. Further, while exploring the problems, students preferred to use a mix of paper-pencil methods and technology. For example, in the Hill cipher method, students performed 2 2 computations by hand and used technology only for lengthy calculations involving 33 or higher order matrices. To summarize, it may be concluded that integrating technology in mathematics teaching and learning at the senior secondary level led to a very satisfying combination of technology use and by-hand skills. The studies indicate that the appropriate use of technol-

ogy can augment learning and provide a rich and motivating environment for exploring mathematics.

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Matrices – Theory and Applications

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Executive Chairman, AMTI

Cayley a French mathematician, discovered matrices in the year 1860. It took another 50 years to be appreciated by engineers. Main usage is in solving a system of equations and a system has as many equations as those of no. of variables. Matrix theory helps to consider an array of many numbers as a single entity, which can be denoted by a single symbol. This helps and carried out in the most compact form. A matrix is a rectangular array of numbers enclosed by parenthesis. Matrices are usually named using upper case letters. Some examples of matrices are:

$$A = \begin{pmatrix} -4 & 3 & -6 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 5 & -2 \\ 3 & -7 \end{pmatrix} \quad C = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 7 & -4 \\ 0 & 1 & -3 \end{pmatrix}$$

The individual numbers in a matrix are called the elements. A horizontal arrangement of the numbers in a matrix is called a row. A vertical arrangement of the numbers in a matrix is called a column. The number of rows and the number of columns in a matrix is called the dimensions of a matrix. However, square matrices (any matrix that has the same number of rows as it has columns) have wider applications. Matrix theory is a part of Linear Algebra.

When Matrix theory is useful? In the study of a system of

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vibrating spring, it might be observed that shark observers are used in trucks, car, bikes etc. The vibration of this system is to be studied so that travel can be made comfortable. Theory of Vibrations is a separate subject for mechanical engineering students. FEAST (Finite element analysis of structures) is very useful software for construction engineers. Theory of electric circuit is also an application of matrix theory.
Matrix is applied to several branches of science, as well as different mathematical disciplines.

1. Engineering: Forces in a bridge or truss
   A typical statics problem is represented by the following: There are 3 unknown forces F_1, F_2 and F_3. From

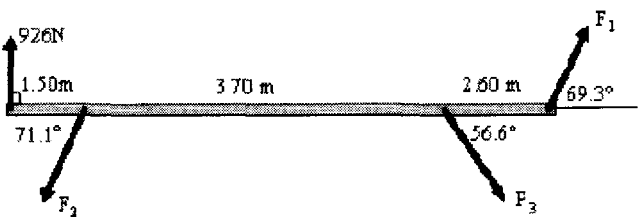

Figure 10: Forces on a bridge

   the diagram, we can obtain 3 equations involving the 3 unknowns and then solve the system using matrix operations.

2. Electronics: The circuit in 11 has 7 unknown currents, $I_1, I_2, \ldots, I_7$. We can set up 7 equations in the unknown currents and use matrices to solve and find the values of $I_1, I_2, \ldots, I_7$.

3. Other applications: Matrices are also used to solve

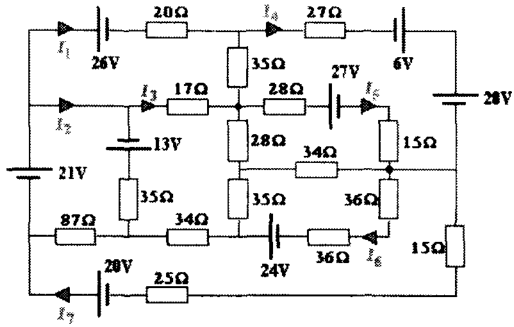

Figure 11: Circuit Application

problems in:

(a) Genetics (working out selection processes)

(b) Chemistry (finding quantities in a chemical solution)

(c) Economics (study of stock market trends, optimization of profit and minimization of loss)

(d) Computer Graphics: Computer graphics software uses matrix mathematics to process linear transformations and render images. A square matrix, one with exactly as many rows as columns, can represent a linear transformation of a geometric object. For example, in the Cartesian XY plane, the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

reflects an object in the vertical Y axis. In a video game, this would render the upside-down mirror image of a castle reflected in a lake.

If the video game has curved reflecting surfaces, such as a shiny silver goblet, the linear transformation matrix would be more complicated, to stretch or shrink the reflection.

Singular Matrix

Singular matrix is a square matrix whose determinant is equal to zero.

Example: $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

Non singular matrix

A square matrix that has an inverse is called non singular. These are also called regular matrices. A square matrix is non singular if and only if its determinant is non zero.

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

Characteristic equation of a matrix

If A is a square matrix and I a unit matrix of the same order as A. If λ is a scalar such that $A - \lambda I$ is singular we call λ an eigen value of A. The eigen value of A are given by the equation $|A - xI| = 0$, where $|\cdot|$ denotes the determinant. This equation is called the characteristic equation of the matrix A. The roots of the characteristic equation are the eigen values of the matrix and are extremely important in the analysis of many problems in Mathematics and Physics.

Example:

1. Find the characteristic equation of $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$
The characteristic equation is given by

$$\begin{vmatrix} 1-x & 1 \\ 3 & -1-x \end{vmatrix} = (1-x)(-1-x) - 3 = 0$$

and hence the characteristic equation is $x^2 - 4 = 0$.

Cayley Hamilton Theorem Every square matrix satisfies its own characteristic equation.

Applications

- To calculate the positive integral powers of a square matrix
- To find the inverse of a square matrix

Examples

1. Using Cayley Hamilton theorem, find the inverse of $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$
   The characteristic equation is

$$\begin{vmatrix} 1-x & 2 \\ 4 & 3-x \end{vmatrix} = x^2 - 4x - 5 = 0$$

Since A satisfies its characteristic equation, we have

$$A^2 - 4A - 5I = 0$$

Multiplying the above by A^{-1}, we get

$$A - 4I - 5A^{-1} = 0$$

and hence

$$A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$$

2. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ and hence find its inverse.

Solution It is easy to see that the characteristic equation of the given matrix is $x^3 - 3x^2 - x + 9 = 0$

$$A^3 - 3A^2 - A + 9I = \begin{pmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{pmatrix} + \begin{pmatrix} -12 & 9 & -18 \\ -9 & -6 & -12 \\ 0 & 6 & -15 \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & 0 & -3 \\ -2 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also

$$A^{-1} = \frac{1}{9}(-A^2 + 3A - I)$$

$$= \begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix}$$

Eigen Values and eigen vectors

If λ is an eigen value of a square matrix A, a vector $v \neq 0$ such that $Av = \lambda v$ is called an eigen vector corresponding to the eigen value λ. The eigenvalues are used to determine the

natural frequencies (or eigen frequencies) of vibration, and the eigenvectors determine the shapes of these vibrational modes.

In Structural Engineering:

Most structures from buildings to bridges have a natural frequency of vibration. It means all these structures have their own system of eigen vibrations and eigen frequencies. Now external forces like wind and earthquake may cause these structures to undergo vibrations. In case the frequency of these vibrations becomes equal to the natural frequencies of these structures, vibrations with large amplitudes are set up. It is a phenomena called Resonance. This can lead to the collapse of the structure by a process called aeroelastic flutter. One very famous example of the collapse of a structure due to this phenomena is the Tacoma Narrows Bridge (1940) in which the wind provided an external periodic frequency that matched the bridge's natural structural frequency. So vibration analysis of these structures are done at the time of their design using eigenvalues and eigenvectors.

Eigenvalues can also be used to test for cracks or deformities in structural components used for construction. When a beam is struck, its natural frequencies (eigenvalues) can be heard or measured. If the beam "rings," then it is not flawed. A dull sound will result from a flawed beam because the flaw causes the eigenvalues to change. Sensitive machines can be used to "see" and "hear" eigenvalues more precisely.

The eigenvalues can also be used to determine if a structure has deformed under the application of a particular force. Eigenvalues for the structure are measured before and after the

application of force. If a change in the eigenvalues is observed, it means the structure has undergone deformation. This is just one of the fields that make practical use of the eigenvalues of a matrix.

Eigenvalues were used by Claude Shannon to determine the theoretical limit to how much information can be transmitted through a communication medium like your telephone line or through the air. This is done by calculating the eigenvectors and eigenvalues the communication channel (expressed a matrix), and then water filling on the eigen values. The eigenvalues are then, in essence, the gains of the fundamental modes of the channel, which themselves are captured by the eigenvectors.

The eigenvalues and eigenvectors of a matrix are often used in the analysis of financial data and are integral in extracting useful information from the raw data. They can be used for predicting stock prices and analyzing correlations between various stocks, corresponding to different companies. They can be used for analyzing risks. There is a branch of Mathematics, known as Random Matrix Theory, which deals with properties of eigenvalues and eigenvectors, that has extensive applications in Finance, Risk Management, Meteorological studies, Nuclear Physics, etc.

Thus one can see that Matrix Algebra is fundamental to several applications in Science, Engineering and Economics.

The Man and the Genius : Ramanujan

Dr. Devbhadra V. Shah
Secretary, Gujarat Ganit Mandal, Surat

Famous Indian genius, Srinivasa Ramanujan was a largely self-taught pure mathematician. Hindered by poverty and ill-health, his highly original work has considerably enriched number theory and, more recently, Physics. Srinivasa Ramanujans extraordinary contributions to mathematical analysis, number theory, infinite series and continued fractions are unparalleled. Ramanujans work has a truly transformative effect on modern mathematics and continues to do so as we understand further lines from his letters and notebooks. This talk The Man and the Genius : Ramanujan focused mostly on the life span of Ramanujan and provide an account of his amazing, albeit short life. In a sense it can be regarded as a photo-biography of Ramanujan. In this lecture, some of the studies of Ramanujan which are most accessible to the general public were presented.

Prof R.C.Gupta Endowment *Lecture in the* 51^{st} Annual *Conference of the* Association of Mathematics Teachers of India (AMTI)

RMO 1998 problem – A Correction

The following problem is from RMO 1998:

The sum of twenty integers (not necessarily distinct) $d_1, d_2, \ldots, d_{20}$ is 801. What is the least possible value of the LCM of $d_1, d_2, \ldots, d_{20}$?

The solution that appeared in an earlier issue of this journal was incorrect. The correct solution is given below:

Since $40 \times 20 = 800 < 801$, the LCM can not be less than or equal to 40. If the LCM is 41, this number being a prime, we need the twenty numbers to be equal to 41 or 1. This is clearly impossible. Thus it must be 42 or more. For 42, we write 801 as sum of nineteen 42s and the sum of these is 798 and hence $801 = 19 \times 42 + 3$. Since 3 divides 42, the LCM of these numbers is 42.

Going Digital – A Proposal

During the General Body meeting of AMTI held during the 52nd Annual Conference at Bangalore, the General Secretary expressed concern about the raising printing and postage costs for the journal The Mathematics Teacher. It was suggested that we can consider providing e-copy of the journal to those members who opt for the same, thereby reducing costs. If you are a member of AMTI receiving print copy of the Mathematics Teacher currently and wish to receive only e-copy, you can send an email to AMTI at support@amtionline.com and share your email id. A pdf file of the journal will be sent to you. Thank you for helping AMTI in reducing costs.

ASSOCIATION ACTIVITIES

1. The 52nd Conference of the association took place at Jain Heritage School, Bangalore on 27th, 28th and 29th December 2017.There were 182 registrations and 168 attended. The Host school earlier conducted an inter school maths Exhibition, selected ten best and exhibited, as part of the conference exhibition to which delegates exhibits also were added to occupy 3 rooms. It was inaugurated by Sri Gadhadhar Mishra, one of the invited guests, sharp at 10 am on 27th. This was followed by the inaugural function starting with prayer, release of souvenir and inaugural address by Sri Prabhu of I.I.I.T. President of AMTI Prof. I.K. Rana gave his presidential address after which Prof. A.Narasinga Rao memorial lecture was delivered by Dr. S. Muralidharan on how to motivate the learner interested in learning.

   There were two sessions each of paper presentation by delegate teachers and students whereas the latter was held in three parallel sessions to accommodate more aspirants.

   The first day ended with entertainment arranged by the school including yaksha gana bharatanatyam besides fusion music and dance.

   The second day of the conference started with a written quiz for the interested delegates and it was a sight to enjoy with more than a hundred teachers and students writing the same with total involvement. Then Prof. Roddam Narashima former Director, Aerospace

Laboratory, delivered Prof. P.L. Bhatnagar memorial lecture on some aspects of trigonometry from Aryabhata and other Indian astronomers. Prof. V.G. Tikekar chaired the session. The talk on the theme of the conference – Challenges of mathematics teaching today, was then delivered by Dr. M. Palanivasan after which lunch was announced.

In the post lunch session Dr. R. Sivaraman delivered Prof. R.C. Gupta endowment lecture on evolution of equations touching upon contributions from across the globe over the years on the topic.

The panel discussion on the theme followed with Dr. R. Shanti, Smt Veena Dhingra, Smt. Vaishali Lakhoni and Sri G.Gnanasundaram as panelists moderated by Sri. R. Athmaraman. It was well received.

Then the General body of the AMTI met with 37 members, elected the president and executive committee members to function from 1st April 2018. Sri.M.Mahadeven who first as treasurer from 1994, became secretary and then general secretary as per amended byelaws, requested the GB to relieve him w.e.f 31.3.2018. Dr. R. Shanthi was elected as his successor.

On the third day the much awaited quiz with 4 teams of 16 participants, selected on the basis of their performance in written quiz, took place with Sri R. Athmaraman as the quiz master ably assisted by Sri Lakshmi Narayanan and Dr. R. Shanthi.

The post lunch session started with short communication and open session wherein as many as eight participants

shared their opinion including Sri Venkatesa Murthy indicating some aspects of Bhaskara II.

In the valedictory function starting sharp at 2p.m with prayer, five delegates including one student expressed their experiences in the conference.

Prof. V. Krishnamurthy, who had been an earlier executive Vice President of AMTI and who has donated sizable amount as endowment in honour of his professor Sri V.Ganapathy Iyer to award honorarium to invited /theme talk and quiz prizes, spoke motivating the delegates on some matters of math learning. Particular mention may be made on PLEASE with P-Precision, L-Logic, E-Essential, A- Abstract, S-Symbols and E-Evaluation playing a vital rule in maths. Dr.B.T. Venkatesh, controller of exams Jain University and Director Jain PU and Degree programmes was the chief guest who addressed the delegates on several significant aspects of education. The conference ended with vote of thanks and parting address by the General Secretary followed by the National Anthem.

2. The annual workshop in 2018-19 is to take place from May 2nd to 26th in four batches as displayed in our website amtionline.com. We have received proposals, to organize workshops in Bangalore for students and teachers in May and June 2018. Several of our resource persons are invited for such workshops in Trichy, Tuticorin, Kallakurichi etc., by schools and publishers.

3. NMTC 50 announcement is getting ready. Tentative dates for screening test is 1st September 2018 and final

will be on 3rd November 2018.

4. Our computer lab oriented workshop for 15 children took place on 22.09.2017 which was well received.

5. We are planning online orientation sessions for Olympiads.
   More details will appear in our website.

6. As General Secretary and publisher of this journal, I take leave of all members thanking each and everyone for the understanding cooperation extended to me during my term. It is hoped, the same, if not better, will be extended to my successors also to make the name by AMTI reverberating everywhere for math learning and excellence.

Scenes from 52nd conference at Bangalore 27th to 29th December 2017


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